Partial Solution Set, Leon §5.5

5.5.2b We have $\mathbf{u}_1 = \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}} \frac{-4}{3\sqrt{2}}\right)^T$, $\mathbf{u}_2 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)^T$, and $\mathbf{u}_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T$. Let $\mathbf{x} = (1,1,1)^T$. Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, and use Parseval's formula to compute $||\mathbf{x}||$.

Solution: We know from part (a) that $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ is an orthonormal basis for R^3. By Theorem 5.5.2, we know that

$$\begin{aligned}\mathbf{x} &= (\mathbf{x}^T\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x}^T\mathbf{u}_2)\mathbf{u}_2 + (\mathbf{x}^T\mathbf{u}_3)\mathbf{u}_3 \\ &= \frac{-2}{3\sqrt{2}}\mathbf{u}_1 + \frac{5}{3}\mathbf{u}_2 + 0\mathbf{u}_3 \\ &= \frac{-2}{3\sqrt{2}}\mathbf{u}_1 + \frac{5}{3}\mathbf{u}_2\end{aligned}$$

By Parseval's formula, $||\mathbf{x}|| = \left(\frac{4}{18} + \frac{25}{9}\right)^{1/2} = \sqrt{3}$.

5.5.3 We are given S, the subspace spanned by $\mathbf{u}_2$ and $\mathbf{u}_3$ of the preceding exercise, and $\mathbf{x} = (1,2,2)^T$. We are to find the projection $\mathbf{p}$ of $\mathbf{x}$ onto S, and to verify that $\mathbf{p} - \mathbf{x} \in S^\perp$.

Solution: The projection is

$$\begin{aligned}\mathbf{p} &= (\mathbf{x}^T\mathbf{u}_2)\mathbf{u}_2 + (\mathbf{x}^T\mathbf{u}_3)\mathbf{u}_3 \\ &= \frac{8}{3}\mathbf{u}_2 - \frac{1}{\sqrt{2}}\mathbf{u}_3 \\ &= \left(\frac{23}{18}, \frac{41}{18}, \frac{8}{9}\right)^T\end{aligned}$$

So $\mathbf{p} - \mathbf{x} = (\frac{5}{18}, \frac{5}{18}, -\frac{10}{9})^T$. It is easy to show that $\mathbf{p} - \mathbf{x} \in S^\perp$, by showing that it is orthogonal to each of $\mathbf{u}_2, \mathbf{u}_3$.

Note: A close look at the computation by which the projection was obtained is consistent with the observation (Corollary 5.5.9) that the projection operator is UU^T, where U in this case is the matrix whose columns are $\mathbf{u}_1$ and $\mathbf{u}_2$.

5.5.5 Let $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis for R^2, and let $\mathbf{u}$ be a unit vector in R^2. If $\mathbf{u}^T\mathbf{u}_1 = \frac{1}{2}$, determine the value of $|\mathbf{u}^T\mathbf{u}_2|$.

Solution: Since $\mathbf{u}$ is a unit vector, and since $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis for R^2, then by Parseval's formula we know that $(\mathbf{u}^T\mathbf{u}_1)^2 + (\mathbf{u}^T\mathbf{u}_2)^2 = 1$. Given $\mathbf{u}^T\mathbf{u}_1 = \frac{1}{2}$, it follows that $(\mathbf{u}^T\mathbf{u}_2)^2 = \frac{3}{4}$, so $|\mathbf{u}^T\mathbf{u}_2| = \frac{\sqrt{3}}{2}$.

5.5.6 Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V, and let

$$\mathbf{u} = \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3 \text{ and } \mathbf{v} = \mathbf{u}_1 + 7\mathbf{u}_3.$$

Determine the value of each of the following:

(a) $\langle \mathbf{u}, \mathbf{v} \rangle$

(b) $||\mathbf{u}||$ and $||\mathbf{v}||$

(c) The angle θ between $\mathbf{u}$ and $\mathbf{v}$.

Solution:

(a) By Corollary 5.5.3, $\langle \mathbf{u}, \mathbf{v} \rangle = 1 + 0 + 14 = 15$.

(b) By Parseval's formula, $||\mathbf{u}|| = (1 + 4 + 4)^{1/2} = 3$, and $||\mathbf{v}|| = (1 + 0 + 49)^{1/2} = 5\sqrt{2}$.

(c) Using our results from (a) and (b), we have

$$\theta = \arccos \frac{15}{15\sqrt{2}} = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

5.5.14 Let $\mathbf{u}$ be a unit vector in $\mathbf{R}^n$, and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is both orthogonal and symmetric and hence is its own inverse.

Proof: The symmetry of H follows from the symmetry of I and the symmetry of $\mathbf{u}\mathbf{u}^T$, i.e., $\left(\mathbf{u}\mathbf{u}^T\right)^T = \mathbf{u}^{T^T}\mathbf{u}^T = \mathbf{u}\mathbf{u}^T$, along with the fact that the sum of symmetric matrices is symmetric. To show that H is orthogonal, we show that $H^T H = I$:

$$\begin{aligned} H^T H &= ((I - 2\mathbf{u}\mathbf{u}^T)^T(I - 2\mathbf{u}\mathbf{u}^T) \\ &= I^T I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T \\ &= I^2 - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\left(\mathbf{u}^T\mathbf{u}\right)\mathbf{u}^T \\ &= I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T \\ &= I. \end{aligned}$$

But if H is both orthogonal and symmetric, then $H^{-1} = H^T = H$. $\square$

5.5.17 Show that if U is $n \times n$ orthogonal, then $\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T = I$.

Solution: Since U is orthogonal, then (see exercise 10 in this section) so is U^T, i.e., $UU^T = I$. But then

$$\begin{aligned} I &= UU^T \\ &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \\ &= \mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T, \end{aligned}$$

and the result follows. □

5.5.19.b.ii Let $A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

Solve the least squares problem $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (1, 2, 3, 4)^T$.

Solution: Since the columns of A constitute an orthonormal set, it follows that $A^TA = I$, and the normal equations reduce to

$$\hat{\mathbf{x}} = A^T\mathbf{b} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$